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Over-reflection and shear instability in a shallow-water model

By SHIN-ICHI TAKEHIRO AND YOSHI-YUKI HAYASHI

Department of Earth and Planetary Physics, University of Tokyo, Bunkyo, Tokyo 113, Japan

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The characteristics of shallow-water waves in a linear shear flow are studied, and the relationship between waves and unstable modes is examined. Numerical integration of the linear shallow-water equations shows that over-reflection occurs when a wave packet is incident at the turning surface. This phenomenon can be explained by the conservation of momentum as discussed by Acheson (1976). The unstable modes of linear shear flow in a shallow water found by Satomura (1981) are described in terms of the properties of wave propagation as proposed by Lindzen and others. Ripas's (1983) theorem, which is the sufficient condition for stability of flows in shallow water, is also related to the wave geometry. The Orr mechanism, which is proposed by Lindzen (1988) as the primary mechanism of wave amplification, cannot explain the over-reflection of shallow-water waves. The amplification of these waves occurs in the opposite sense to that of Orr's solution.

1. Introduction

Since the experiments by Reynolds and theoretical studies by Rayleigh at the end of 19th century, the stability of shear flows has been investigated intensely. In particular, the linear stability of two-dimensional parallel shear flow has interested workers in fluid dynamics, because of its mathematical simplicity and usefulness. The linear stability of a shear flow is studied as an eigenvalue problem. If we get at least one eigenvalue (phase speed c) whose imaginary part is positive, the flow field is unstable. Moreover, the structure of the eigenfunction that has the maximum growth rate is believed to appear in the transition from the unstable basic state to another state.

From the mathematical solution of the eigenvalue problem, however, it is not easy to obtain an intuitive image of the reason why the flow field should be unstable (Lindzen 1988). It is difficult to determine the stability of the flow at first sight without using mathematical tools. We have to solve the eigenvalue problem repeatedly for each flow configuration. Even after one obtains an unstable mode, all one can show is that the unstable mode has a consistent structure for its growth. One cannot obtain any physical answer as to why the disturbances should be amplified. Moreover, the stability often varies dramatically with a slight change of the basic state or boundary conditions. Sometimes, the results are even opposed to intuition. For example, a linear shear flow of an unstratified, incompressible fluid has no unstable mode (Orr 1907). However, for the stratified case, even a linear shear flow can be unstable (Howard & Maslowe 1973). This is opposite to the belief that stratification stabilizes fluids. A linear shear flow can also be unstable in shallow water (Satomura 1981). Solving eigenvalue problems does not explain the reason for this variety of stability of linear shear flows.

In some cases, it is possible to derive conditions for the stability of general flow configuration. These are called integral theorems (Fjørtoft 1950), since they are obtained from the equations integrated over the fluid. However, these theorems, similar to the solutions of eigenvalue problems, do not give us an intuitive image of instability. For example, the expression of Rayleigh's (1880) inflexion point theorem does not directly answer the question of why an inflexion point is necessary for instability.†

In order to improve the physical understanding of shear instability, unstable situations have been described by properties of wave propagation in shear flows. The merit of this description is that one can describe the characteristics of flows locally and intuitively because waves are local solutions. It is then possible to consider the characteristics of flows and boundary conditions separately, which should be handled at the same moment in the eigenvalue problems. Moreover, one can discuss the stability of different systems in terms of the same concept by describing the instability using the general term 'wave'.

The remarkable phenomenon in describing instability of shear flows with properties of wave propagation is over-reflection. The occurrence of over-reflection was discovered by Miles (1957) for sound waves, and the situations where it occurs are summarized by Acheson (1976). He explained over-reflection by the conservation of wave action. However, the relationship between the occurrence of over-reflection and the existence of unstable modes is ambiguous in the understanding of people at that time, including Acheson. In fact, Acheson (1976) computed steady over-reflection solutions and unstable modes of the vortex sheet model. In his over-reflection solution, both sides of the vortex sheet are wave regions. The unstable modes which can be interpreted by such over-reflection solutions are the modes of the case where a reflecting boundary is located behind the incident wave region. In order to describe unstable modes of the vortex sheet model, over-reflection of 'trapped' Rossby waves in the neighbourhood of the sheet should be considered (e.g. Lindzen & Rosenthal 1983). Therefore, figures 1 and 2 of Acheson (1976) are insufficient.

It was Lindzen and his colleagues who clearly connected over-reflection solutions with unstable modes. The combinations of waves and shear instability they considered are internal gravity waves and stratified shear instability (Rosenthal & Lindzen 1983*a*, *b*; Lindzen & Rosenthal 1983; Lindzen & Barker 1985), Rossby waves and barotropic instability (Lindzen & Tung 1978), Rossby waves and instability of viscous Poiseuille flow (Lindzen & Rambaldi 1986), and Rossby waves and baroclinic instability (Lindzen, Farrel & Tung 1980). They summarized the relationship between the unstable modes and the propagation properties of these waves as follows (Lindzen 1988): the circumstances where over-reflection occurs are described by the structure of the common properties of wave propagation, which is called the wave geometry; the necessary conditions for instability provide the wave geometry for over-reflection; in the solutions of over-reflection, the one that satisfies the 'quantization' becomes an unstable mode. Its growth rate can be evaluated with a 'laser formula'.

Through the investigations of the relationship between over-reflection and unstable modes, Lindzen and his colleagues succeeded in describing a variety of linearly unstable situations in terms of the same concept. However, in the situations

† Taylor (1915) derived the inflexion point theorem from the momentum conservation law. However, many modern textbooks (e.g. Drazin & Reid 1981; Pedlosky 1987) do not follow his derivation. They describe only mathematical proofs of the integral theorems and give almost no physical interpretation to these results.

they considered, it is only the 'vorticity waves' that are related to the unstable modes. In this paper, we investigate an unstable shallow-water layer, in which 'divergent waves' play an important role. The possibility of whether unstable modes can be interpreted as the properties of shallow-water waves will be studied in the same way as the 'vorticity waves'.

Shear instability in shallow water has been investigated by Satomura (1981) for the case of a linear shear flow with no rotation. The stability of shear flows in shallow water on the β-plane was considered by Ripa (1983) and Hayashi & Young (1987). The stability of a compressible fluid, whose equations are the same as those of the shallow-water case, was studied by Narayan, Goldreich & Goodman (1987). Here, in order to have only pure divergent shallow-water waves, we consider a linear shear flow with no rotation, in which the 'vorticity waves', i.e. Rossby waves do not appear.

In §2, the linearized equations of the shallow-water system are described. In §3, the behaviour of shallow-water wave packets in a linear shear flow are investigated as an initial-value problem. In §4, the relationship between unstable modes obtained by Satomura (1981) and steady solutions of over-reflection is examined according to the method of Lindzen. In §5, Ripa's (1983) theorem, the sufficient condition for stability in shallow water, is interpreted for the case of Satomura (1981) with the property of wave propagation.

2. Linearized shallow-water system

Before considering the relationship between the shear instability and the properties of wave propagation in a shallow-water system, we shall list here the basic state and the linearized equations of the system. We start from the non-dimensional shallow-water equations with no rotation:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{Fr^2}\frac{\partial h}{\partial x}, \tag{1}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{Fr^2}\frac{\partial h}{\partial y}, \tag{2}$$

$$\frac{\partial h}{\partial t}+\frac{\partial (hu)}{\partial x}+\frac{\partial (hv)}{\partial y}=0. \tag{3}$$

Fr is the Froude Number expressed as $Fr^2 \equiv U_0^2/gH$, where H is the mean depth, U_0 is a characteristic velocity of the flow, and g is the acceleration due to gravity. Conservation of the potential vorticity gives

$$\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)q=0, \quad q\equiv\frac{(\partial v/\partial x)-(\partial u/\partial y)}{h}. \tag{4}$$

The basic state considered here is a linear shear flow, $u = U(y) = y, v = 0, h = H(=$ constant), in which a horizontal scale L is selected as $L^{-1} = (1/U_0)(\mathrm{d}U/\mathrm{d}y)$, where $\mathrm{d}U/\mathrm{d}y$ is the shear of the basic flow. The linear equations of disturbances u', v', h' for this basic state are

$$\frac{\partial u'}{\partial t}+U\frac{\partial u'}{\partial x}+v'=-\frac{1}{Fr^2}\frac{\partial h'}{\partial x}, \tag{5}$$

$$\frac{\partial v'}{\partial t}+U\frac{\partial v'}{\partial x}=-\frac{1}{Fr^2}\frac{\partial h'}{\partial y}, \tag{6}$$

$$\frac{\partial h'}{\partial t}+U\frac{\partial h'}{\partial x}+\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}=0. \tag{7}$$

The linearized potential vorticity conservation is

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)q'=0, \quad q'\equiv\frac{\partial v'}{\partial x}-\frac{\partial u'}{\partial y}+h'. \tag{8}$$

Eliminating u', v' from (5)–(7), we obtain an equation for h',

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\left\{\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)^2 h'-\frac{1}{Fr^2}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)h'\right\}+\frac{2}{Fr^2}\frac{\partial^2 h'}{\partial x\,\partial y}=0. \tag{9}$$

Since the flow is independent of x, we use the Fourier transform in the x-direction. The equations corresponding to (5)–(8) become

$$\left(\frac{\partial}{\partial t}+\mathrm{i}kU\right)u'+v'=-\frac{\mathrm{i}kh'}{Fr^2}, \tag{10}$$

$$\left(\frac{\partial}{\partial t}+\mathrm{i}kU\right)v'=-\frac{1}{Fr^2}\frac{\partial h'}{\partial y}, \tag{11}$$

$$\left(\frac{\partial}{\partial t}+\mathrm{i}kU\right)h'+\mathrm{i}ku'+\frac{\partial v'}{\partial y}=0, \tag{12}$$

$$\left(\frac{\partial}{\partial t}+\mathrm{i}kU\right)q'=0, \quad q'=\mathrm{i}kv'-\frac{\partial u'}{\partial y}+h'. \tag{13}$$

Since the coefficients are independent of t, solutions of the form $\mathrm{e}^{\mathrm{i}k(x-ct)}$ can be obtained from the following equation corresponding to (9):

$$\frac{\mathrm{d}^2\hat{h}'}{\mathrm{d}y^2}-\frac{2}{U-c}\frac{\mathrm{d}\hat{h}'}{\mathrm{d}y}+k^2\{Fr^2(U-c)^2-1\}\hat{h}'=0, \tag{14}$$

where $h'=\hat{h}'(y)\,\mathrm{e}^{\mathrm{i}k(x-ct)}$. In the following sections, we shall consider the monotonic waves in the x-direction (i.e. $k=$ constant).

3. Over-reflection of wave packets

3.1. *Properties of wave propagation: the WKBJ approximation*

We now consider the behaviour of wave packets in a linear shear flow of the shallow-water system described by (5)–(7). Let us begin with a description utilizing the standard WKBJ approximation method (Whitham 1974). We assume the following form of a wave packet for the surface displacement h':

$$h'=\sum_{n=0}^{\infty}\epsilon^n A_n(x,y,t)\exp\left[\mathrm{i}\frac{\vartheta(x,y,t)}{\epsilon}\right], \tag{15}$$

$$k=\frac{1}{\epsilon}\frac{\partial\vartheta}{\partial x}, \quad l=\frac{1}{\epsilon}\frac{\partial\vartheta}{\partial y}, \quad \omega=-\frac{1}{\epsilon}\frac{\partial\vartheta}{\partial t}, \tag{16}$$

where ϵ is a small parameter, k and l are local wavenumbers in the x- and y-directions, and ω is a local frequency. Substituting (15) into (9), we get the local dispersion relation from the equation of the lowest order in ϵ,

$$\omega=yk\pm\frac{1}{Fr}(k^2+l^2)^{\frac{1}{2}}. \tag{17}$$

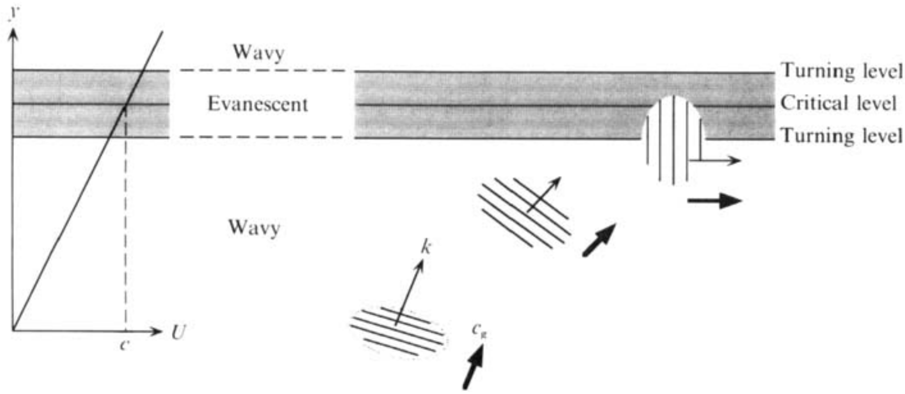

FIGURE 1. Schematic picture of the properties of wave propagation.

The properties of wave propagation obtained from the local dispersion relation (17) are illustrated in figure 1. The position of turning surfaces, $y = y_t$, where $l = 0$, is expressed as

$$y_t = y_c \pm \frac{1}{Fr}, \tag{18}$$

where $y_c \equiv \omega/k$ is the position of the critical level. The region near the critical level sandwiched between two turning surfaces is evanescent, and both regions outside that are wavy. The width of the evanescent region is $2/Fr$.

The wave geometry shown in figure 1 satisfies the geometry for the occurrence of over-reflection summarized by Lindzen (1988): the critical level exists at $y = y_c$; there is another wave region (wave sink) on the opposite side of the incident wave region with respect to the critical level; the waves can reach the critical level because it is in an evanescent region.

The packet can reach the turning surface in a finite time, though the group velocity of a packet vanishes as it approaches there. However, in the framework of the WKBJ approximation, we cannot predict the packets behaviour around the turning surface, because it is not appropriate there.

3.2. *Initial-value problem: over-reflection of wave packets*

Since we cannot predict the behaviour of wave packets after reaching the turning surface, we carry out the time integration of the linear equations (10), (12) and (13) (see Appendix A). The initial value is the following Gaussian-type wave packet placed at $y = y_0$ which is far from the critical level:

$$h' = \exp\left[-\left(\frac{y-y_0}{a}\right)^2 + \mathrm{i}ly\right], \tag{19}$$

$$u' = -\frac{1}{Fr^2(y-c)}h', \tag{20}$$

$$q' = 0, \tag{21}$$

where $l = k(Fr^2(c-y_0)^2-1)^{\frac{1}{2}}$, $c = \omega/k$ is a phase speed and a is an e-folding width of the packet. The results of integrations with three values of Fr/k are shown in figure 2. The wave packets reach the turning surfaces as expected from the WKBJ approximation. However, the behaviour after that varies with k/Fr. At the turning

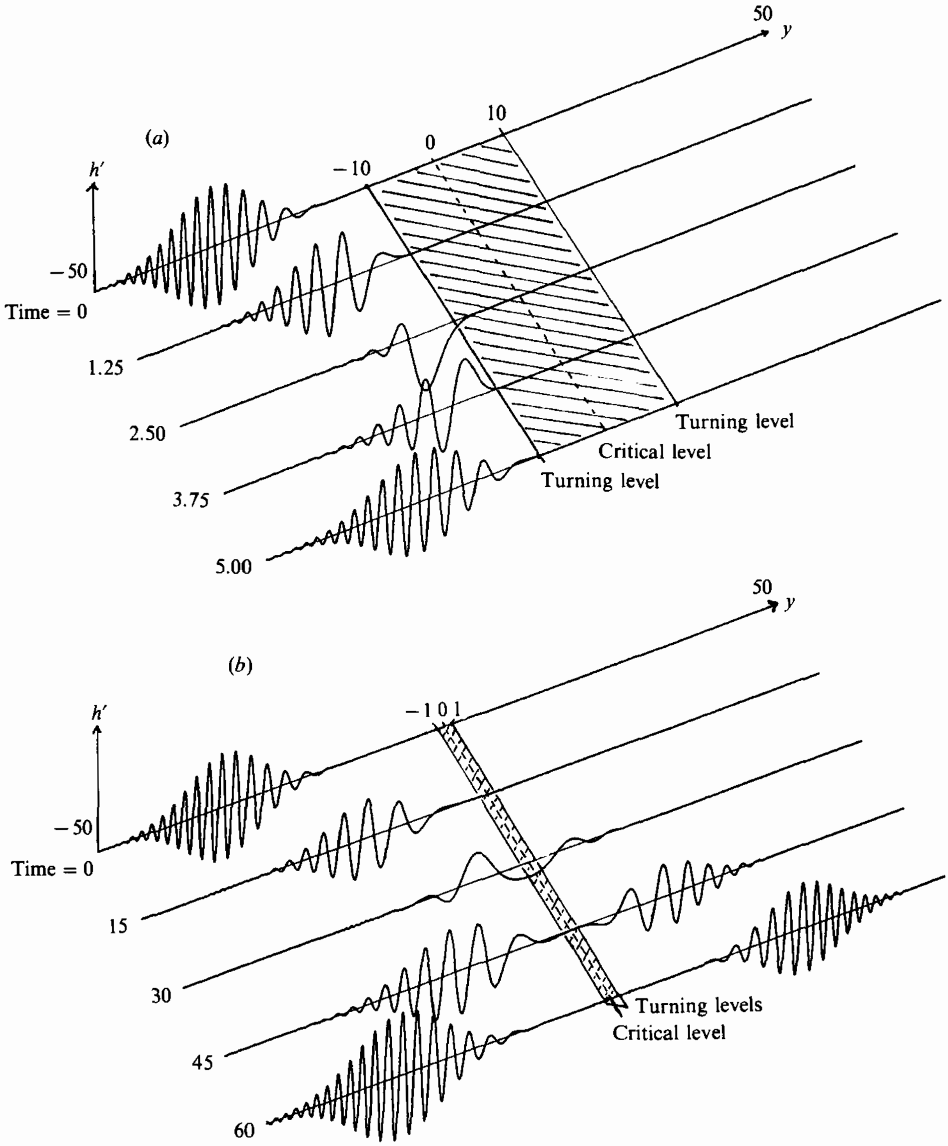

FIGURE 2 (*a*, *b*). For caption see facing page.

surface, the wave packets try to penetrate the evanescent region. The depth of penetration is proportional to $1/k$. When the width of the evanescent region $2/Fr$ is greater than this depth, the packets can hardly tunnel through this region, which results in normal reflection (figure 2*a*). When the width is less, on the other hand, the packets can easily tunnel through the evanescent region, and the incident wave packets are divided into reflected and transmitted packets (figure 2*b*, *c*). Note that over-reflection occurs in this case. The amplitudes of the reflected wave packets become larger than those of incident ones.

3.3 *Description of over-reflection in terms of disturbance momentum*

Over-reflection can be described using the conservation of momentum in the x-direction. We define a disturbance momentum after Hayashi & Young (1987).

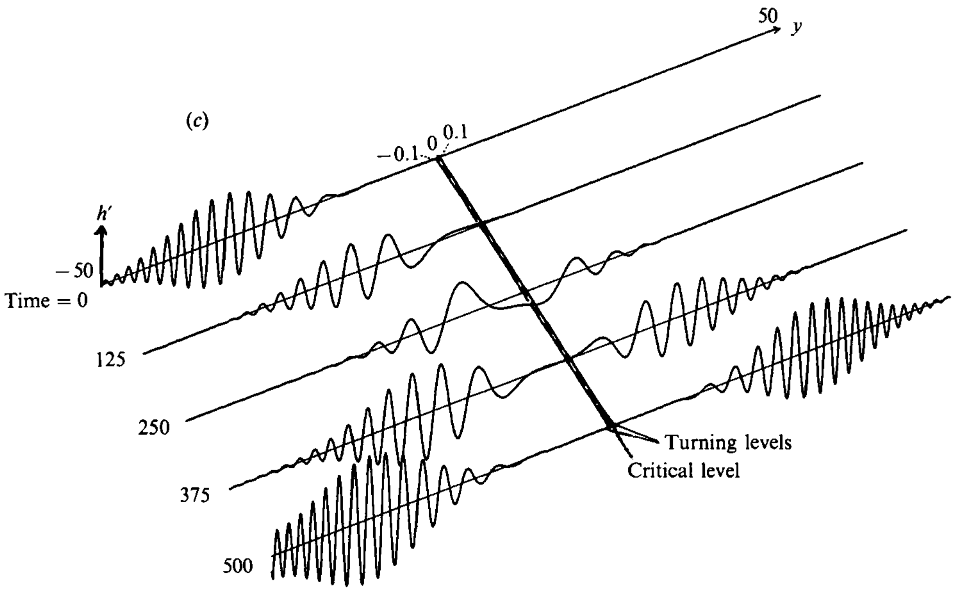

FIGURE 2. Over-reflection of shallow-water wave packets. Time series of surface displacement are plotted. The parameters of the initial packets are $y_0 = -75$, $a = 7.5$, $\omega = 0$. The shaded areas indicate the evanescent region. (*a*) $Fr = 0.1$, $k = 1.0$; (*b*) $Fr = 1$, $k = 0.1$; (*c*) $Fr = 10$, $k = 0.01$.

Upon adding (1) $\times h$ and (3) $\times u$ and averaging in the x-direction, we get the momentum conservation equation,

$$\frac{\partial(\overline{hu})}{\partial t}+\frac{\partial(\overline{huv})}{\partial y}=0, \tag{22}$$

where $\overline{(\;)}$ denotes a value averaged over x. Integrating (22) over the whole domain with respect to y, we get the conservation equation of total momentum,

$$\frac{\mathrm{d}M}{\mathrm{d}t}=0, \quad M \equiv \int \overline{hu}\,\mathrm{d}y. \tag{23}$$

We split each physical quantity into the basic state quantity, the disturbance quantity of the first order $(\;)'$ and that of a higher order $(\;)^{(2)}$ which is induced by the first-order quantities:

$$u = y+u'+u^{(2)}, \quad v = v'+v^{(2)}, \quad h = H+h'+h^{(2)}. \tag{24}$$

Substituting these equations into (23), we have

$$\frac{\mathrm{d}M_{\mathrm{d}}}{\mathrm{d}t}=0, \tag{25}$$

where

$$M_{\mathrm{d}} \equiv M_1+M_2, \tag{26}$$

$$M_1 \equiv \int (H\overline{u^{(2)}}+U\overline{h^{(2)}})\,\mathrm{d}y, \tag{27}$$

$$M_2 \equiv \int \overline{h'u'}\,\mathrm{d}y. \tag{28}$$

M_{d} is the difference of total momentum between the states where a disturbance exists and where one does not. M_{d} is referred to as the disturbance momentum. Equation (25) states that M_{d} is a conserved quantity.

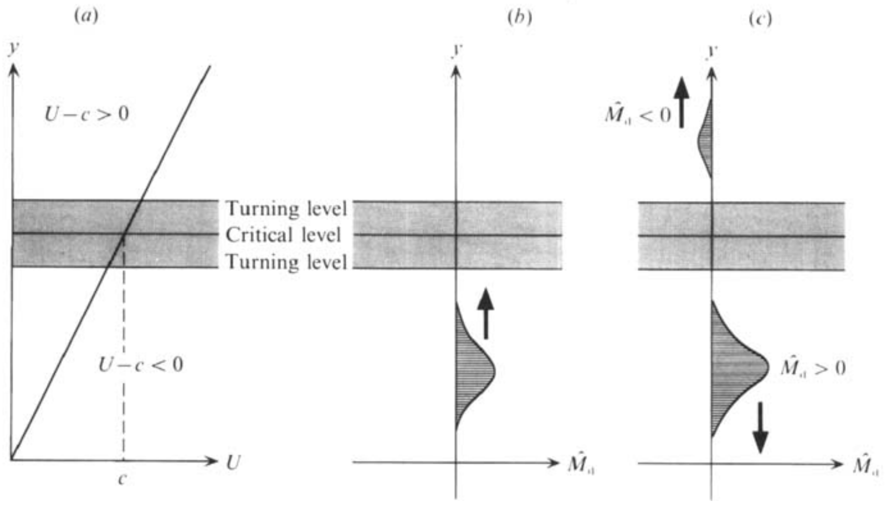

FIGURE 3. Schematic picture of over-reflection.

Integrating (5) $\times h'$ + (7) $\times u'$ with respect to y, we obtain an equation for M_2,

$$\frac{\mathrm{d}M_2}{\mathrm{d}t} = -\int \overline{q'v'}\,\mathrm{d}y. \tag{29}$$

Now, let us consider only the disturbances whose potential vorticity is initially zero. This limitation excludes vorticity waves but retains surface gravity waves. In this case, since we find from (8) that $q' = 0$ at any moment, we have

$$\frac{\mathrm{d}M_2}{\mathrm{d}t} = 0. \tag{30}$$

Substituting this into (25), we get

$$\frac{\mathrm{d}M_1}{\mathrm{d}t} = 0. \tag{31}$$

Since we can choose $M_1 = 0$ at $t = 0$ without losing generality, we can express the disturbance momentum simply by M_2, that is,

$$M_\mathrm{d} = \int \overline{h'u'}\,\mathrm{d}y. \tag{32}$$

When the WKBJ approximation is applicable, (32) can be rewritten as

$$M_\mathrm{d} \sim \int \left(-\frac{1}{Fr^2(y-c)}\right) \overline{h'^2}\,\mathrm{d}y. \tag{33}$$

When the disturbance is in the form of wave packets, the disturbance momentum (32) can be expressed in terms of integration over each wave packet:

$$M_\mathrm{d} = \hat{M}_{\mathrm{d}1} + \hat{M}_{\mathrm{d}2} + \dots, \quad \hat{M}_{\mathrm{d}i} \equiv \int_{i\mathrm{th\,packet}} \overline{h'u'}\,\mathrm{d}y. \tag{34}$$

$\hat{M}_{\mathrm{d}i}$ will be referred to as the disturbance momentum associated with the ith wave packet. The sum of $\hat{M}_{\mathrm{d}i}$ is conserved.

In this situation, schematically illustrated in figure 3(*a*), the sign of $\hat{M}_\mathrm{d}$ can be determined easily from (33), when the packet considered is far from the turning surface. The sign of $\hat{M}_\mathrm{d}$ of a wave packet in the region $y-c > 0$ is negative, while that in the region $y-c < 0$ is positive.

Now, let us consider the evolution of the incident wave packet shown in figure 3(*b*).

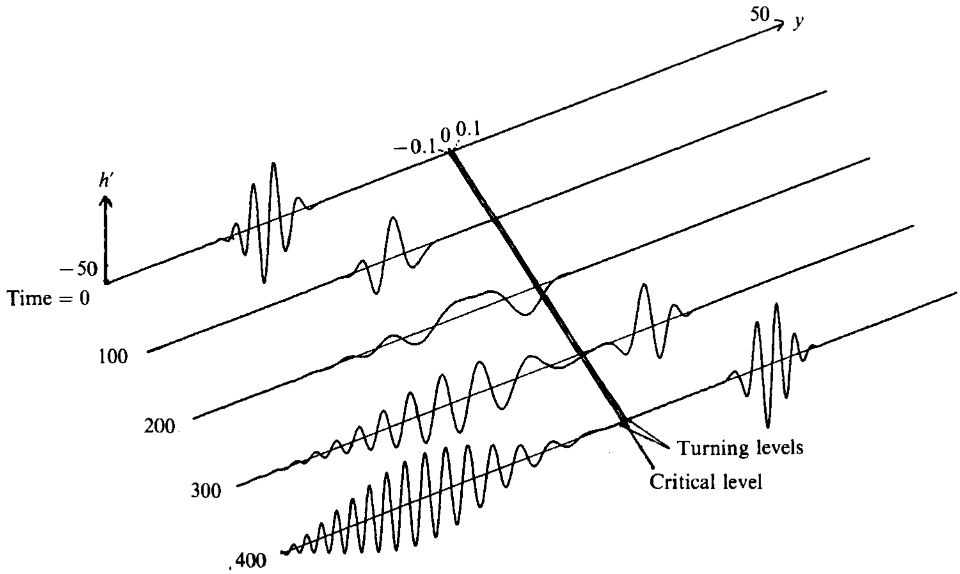

FIGURE 4. An example of a dispersed reflected wave. The parameters are the same as those of figure 2(*b*) except for the initial packet width $a = 2.5$.

The sign of $\hat{M}_d$ of the incident wave packet is positive. The sign of $\hat{M}_d$ of the transmitted wave packet is negative as illustrated in figure 3(*c*). Because the sum of the disturbance momentum of wave packets must be conserved, $\hat{M}_d$ of the reflected wave packet must be larger than that of the incident wave packet. This is the over-reflection.

We have argued that over-reflection of the shallow water waves is the phenomenon by which the incident waves give disturbance momentum of opposite sign to the transmitted wave, resulting in an increase of the disturbance momentum of the reflected wave. This view of over-reflection is the same as the scenario described by Acheson (1976). However, his argument is restricted within the framework of the WKBJ approximation. He did not calculate the evolution of a wave packet.

Acheson's description utilizes a second-order change of the mean flow $u^{(2)}$, while (32) does not. Thus it may seem that the details of the mechanism here are different from that of Acheson. However, the essence of the description of over-reflection is the sign of the conserved quantity of the system, not the existence of $u^{(2)}$. When the conserved quantity of the waves in the incident and reflected region has an opposite sign to that of the waves in the transmission region, we can expect the occurrence of over-reflection because the total value of the conserved quantity does not change before and after incidence. This statement is valid regardless of the kinds of conserved quantity and their description. $u^{(2)}$ is the conserved quantity in the case of Acheson (1976), while $m_d = \overline{h'u'}$ is the conserved quantity in the case of the shallow water presented here (see (32)).

Acheson (1976) used action to explain the mechanism of over-reflection while we used disturbance momentum in the x-direction. This is because m_d can be defined from the ordinary Eulerian form of the equations of motion, while, strictly speaking, action $\mathscr{A}$ should be defined from the Lagrangian of the system (Whitham 1974; Salmon 1988). So long as the WKBJ approximation is valid, m_d coincides with $k\mathscr{A}$. However, in the general situation, m_d does not coincide with $k\mathscr{A}$, although the integrals of them over the whole domain are equal.

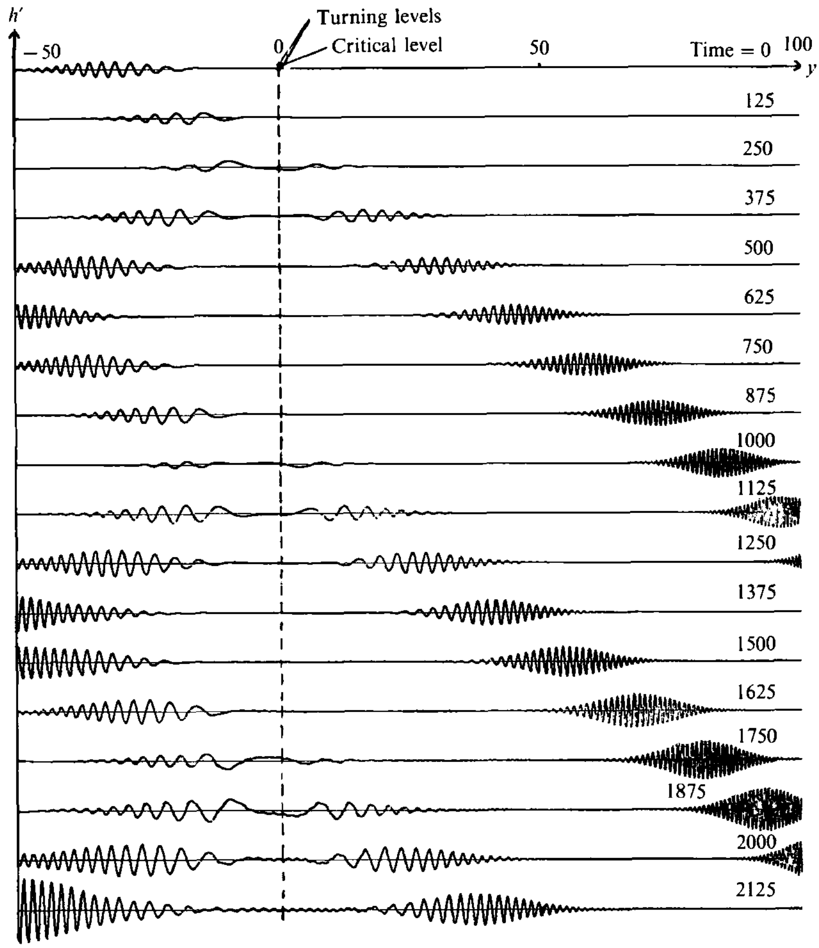

FIGURE 5. Multiple over-reflection of a shallow-water wave. The parameters are the same as those of figure 2(*b*).

The results presented in figure 2 show that the wave geometry for over-reflection summarized by Lindzen (1988) also predicted over-reflection of shallow-water waves. The meaning of another wave region (wave flux sink) here is the region which allows the existence of waves with $\hat{M}_d$ of the opposite sign. There must be such a region, since we have to satisfy the conservation law of momentum. In the shallow-water system considered, an alternative view of the wave geometry for over-reflection is that waves with the same phase speed but with the opposite signed $\hat{M}_d$ can co-exist.

It is worth noting that over-reflection is not always observed as described by Acheson (1976) (figure 4). This is because a wave packet is actually composed of a band of frequencies. When a wave packet is sufficiently monotonic, over-reflection occurs as described by Acheson. The amplitude of a reflected wave is larger than that of an incident one. However, when the frequencies of a packet are widely spread, dispersion of the reflected wave occurs because of the difference of turning surfaces corresponding to each frequency.The amplitude of the reflected wave may not be large, though $\hat{M}_d$ of the reflected wave as a whole is considerably increased. Note that the dispersion of the transmitted wave in figure 4 is small, because the dispersion of the incident side cancels out the dispersion caused by that of the transmitted side.

3.4. *Over-reflection and shear instability*

In terms of the over-reflection of shallow-water wave packets, we can illustrate the instability as follows. Figure 5 shows the numerical solutions of the time evolution of a wave packet when a wall exists behind the incident wave region. The wave packet which is over-reflected at the critical level is reflected again at the wall, and then returns to the critical level, resulting in further over-reflection. Instability of a mode is a phenomenon in which the amplitude of the disturbance increased by repeating this process.

It is also possible to say that instability is caused by the interaction of two disturbances which have opposite-signed $\hat{M}_d$. The amplitude of a disturbance which has, say, positive $\hat{M}_d$ can increase by throwing away negative $\hat{M}_d$ to the other disturbance. This is the description of instability presented by Hayashi & Young (1987).

4. Steady solutions of over-reflection and unstable modes in shallow water

4.1. *Steady solutions of over-reflection*

In this section, we will consider the relationship between unstable modes and over-reflection solutions in shallow-water following the procedure of Lindzen and his colleagues.

We have obtained the steady over-reflection solutions in shallow water after Satomura (1981) (see Appendix B). These solutions coincide with the solutions of sonic waves obtained by Narayan *et al.* (1987). Here, we show a detailed picture from the viewpoint of the relationship between over-reflection solutions and unstable modes. Equation (14) has been solved under the radiation condition in the transmitted wave region. Figure 6 shows an example with parameters $Fr = 7.0$ and $k = 4.0$. The important point in figure 6 is that the amplitude of the incident wave decreases as it approaches the critical level and its wave crests are rotated to be vertical. The amplitudes of the reflected and transmitted waves increase as they depart from the critical level and their wave crests are rotated to be horizontal. This tendency is opposite to that of the Orr mechanism (Lindzen 1988), which is the evolution of a disturbance produced by the vorticity in a shear flow.

Figure 7 shows the reflection and transmission factors evaluated using the asymptotic solutions of (14) at a point far from the critical level. The asymptotic solutions far from the critical level are

$$\tilde{h}'_{\pm} \equiv \left|\frac{Fr\tilde{y}}{k}\right|^{-\frac{1}{2}} \exp\left(\pm \mathrm{i} \int^{\tilde{y}} \left|\frac{Fr\tilde{y}}{k}\right| \mathrm{d}\tilde{y}\right), \tag{35}$$

where $\tilde{y} \equiv ky$. Since the solutions of the scattering problem can be expressed at a point far from the critical level as

$$\tilde{h}' \sim A\tilde{h}'_{-} \quad \text{as} \quad \tilde{y} \rightarrow \infty, \tag{36}$$

$$\tilde{h}' \sim B\tilde{h}'_{+} + C\tilde{h}'_{-} \quad \text{as} \quad \tilde{y} \rightarrow -\infty, \tag{37}$$

we define the reflection and transmission factors R and T in terms of the coefficients A, B, C as

$$R \equiv \left|\frac{C}{B}\right|, \quad T \equiv \left|\frac{A}{B}\right|. \tag{38}$$

Actually, we evaluated the coefficients A, B, C at a distance of two wavelengths away from the turning surface where the WKBJ approximation is sufficiently valid.

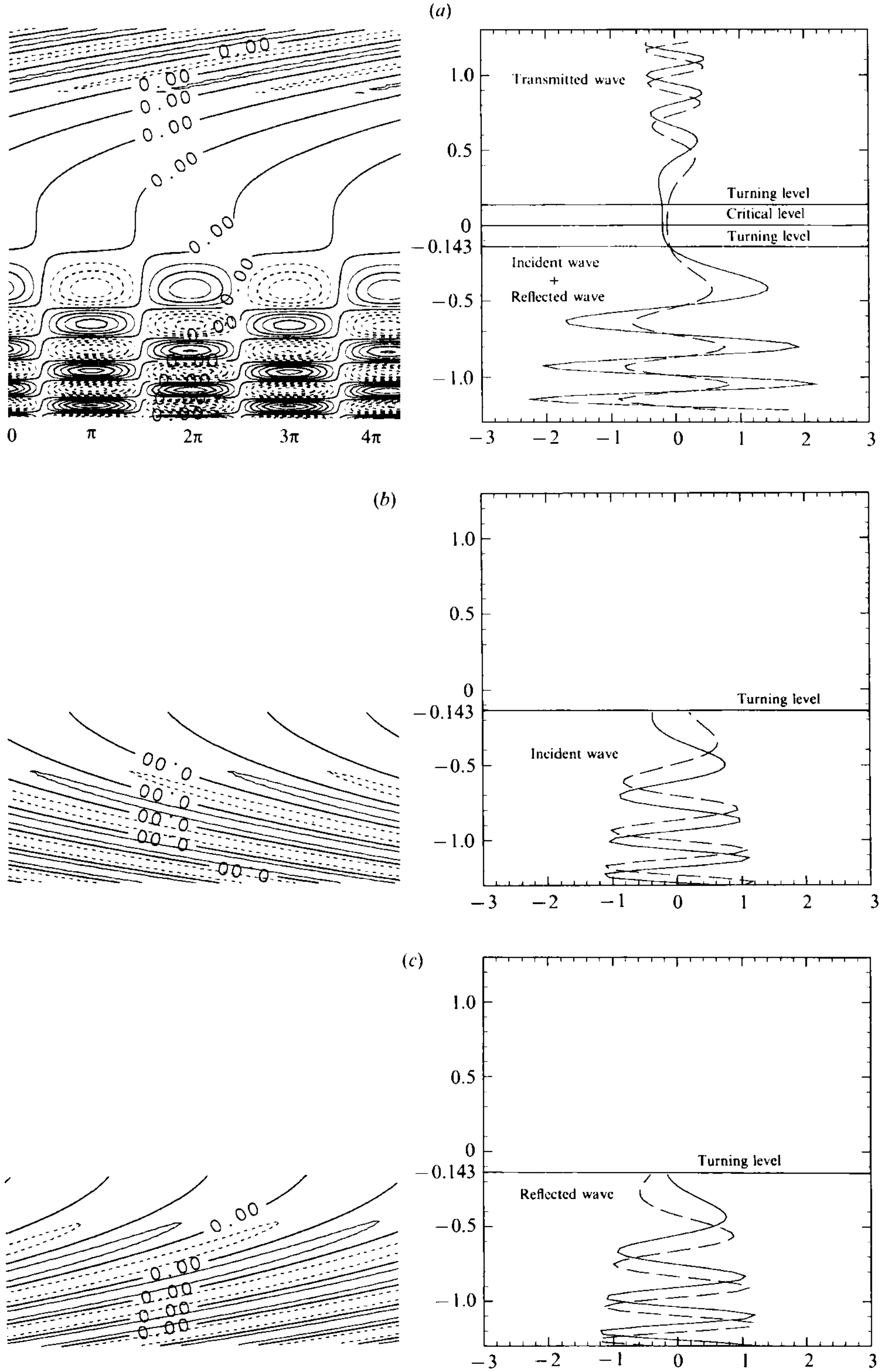

FIGURE 6. (*a*) A steady solution of over-reflection in shallow water ($Fr = 7.0$, $k = 4.0$). Surface displacement h' is shown. (*b*) Incident wave. (*c*) Reflected wave. ——, Re (h'); ---, Im (h').

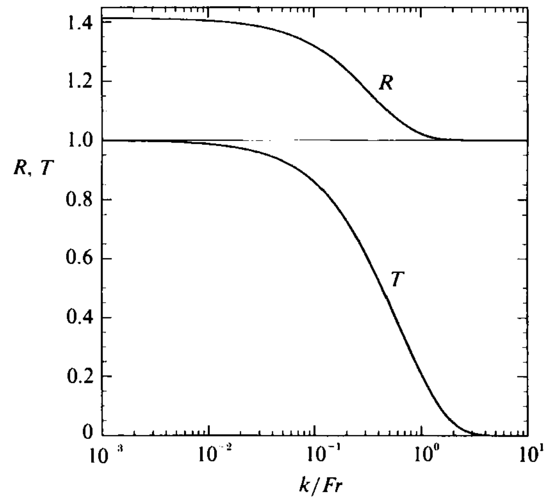

FIGURE 7. Reflection factor (R) and transmission factor (T) of shallow-water waves.

As is shown in figure 7, in the scattering problem of shallow-water waves in a linear shear, over-reflection always occurs for all the parameters. However, the reflection factor is significantly greater than unity only when $k/Fr \lesssim 1$. This corresponds to the results of over-reflection of wave packets presented in §3.2 (figure 2).

Let us describe over-reflection in terms of momentum flux, after Lindzen & Tung (1978). Averaging $h' \times (5) + u' \times (7)$ in the x-direction,

$$\frac{\partial(\overline{h'u'})}{\partial t} = -H^2\overline{q'v'} - \frac{\partial(H\overline{u'v'})}{\partial y}. \tag{39}$$

Since $\overline{q'v'}$ vanishes by the use of the conservation of potential vorticity, (39) is

$$\frac{\partial m_{\mathrm{d}}}{\partial t} = -\frac{\partial(H\overline{u'v'})}{\partial y}. \tag{40}$$

where $m_{\mathrm{d}} \equiv \overline{h'u'}$. Since we consider a steady case, (40) becomes

$$\frac{\partial(H\overline{u'v'})}{\partial y} = 0. \tag{41}$$

The momentum flux, defined as $H\overline{u'v'}$, is constant. Note that there is no momentum flux jump at the critical level in this steady solution (see Appendix B), which is different from the case of vorticity waves studied by Lindzen and his colleagues (for example, Lindzen & Tung 1978). By the application of the WKBJ approximation, $H\overline{u'v'}$ and m_{d} at the point far from the critical level are expressed as

$$H\overline{u'v'} \sim c_{gy}\, m_{\mathrm{d}}, \tag{42}$$

$$m_{\mathrm{d}} = \overline{h'v'} \sim -\frac{1}{Fr^2(U-c)}\,\overline{h'^2}, \tag{43}$$

where $c_{gy} \equiv (1/Fr^2)(l/\omega)$ is the group velocity in the y-direction. Since $c_{gy} > 0$ and $m_{\mathrm{d}} < 0$ for a transmitted wave, $H\overline{u'v'}$ is negative. For an incident wave, since $c_{gy} > 0$ and $m_{\mathrm{d}} > 0$, $H\overline{u'v'}$ is positive. For a reflected wave, since $c_{gy} < 0$ and $m_{\mathrm{d}} > 0$, $H\overline{u'v'}$ is negative. Because $H\overline{u'v'}$ must be constant overall, the amplitude of the reflected wave becomes larger than that of the incident wave (figure 8). This is referred to as over-reflection.

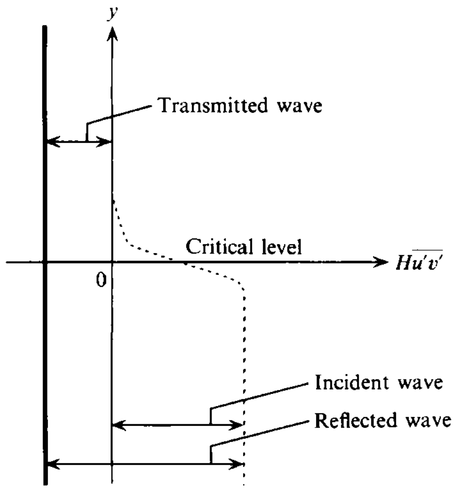

FIGURE 8. The description of steady over-reflection solutions with momentum flux.

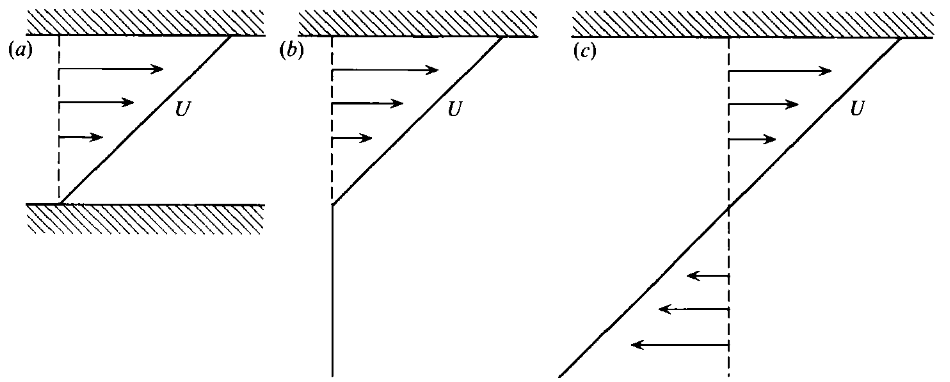

FIGURE 9. Basic flow configurations: (a) case I of Satomura (1981); (b) case II of Satomura (1981); (c) the situation we consider.

4.2. *Quantization and laser formula*

We examine the applicability of quantization and the laser formula of Lindzen and his colleagues in describing unstable modes in a shallow water. Figures 9(*a*) and 9(*b*) show the unstable situation considered by Satomura (1981). Here, we shall compare the growth rates of unstable modes for Case II (figure 9*b*) of Satomura (1981) with those calculated by the quantization and laser formula for the situation of figure 9(*c*).

The growth rate is estimated from the following equation in terms of the reflection factor R (Lindzen 1988):

$$kc_\mathrm{i} = \frac{\ln R}{2\tau}, \tag{44}$$

where τ is the time for a wave to propagate from the reflecting wall ($y = y_\mathrm{b}$) to the turning surface ($y = y_\mathrm{t}$),

$$\tau = \int_{y_b}^{y_\mathrm{t}} \frac{\mathrm{d}y}{c_{gy}}.$$

The condition of quantization is selected as $\mathrm{Re}\,[\mathrm{d}h'/\mathrm{d}y] = 0$ at the boundary.

Figure 10 shows the result. The grey lines show the corresponding dispersion relations and growth rates of Satomura (1981) and the black lines show the estimates

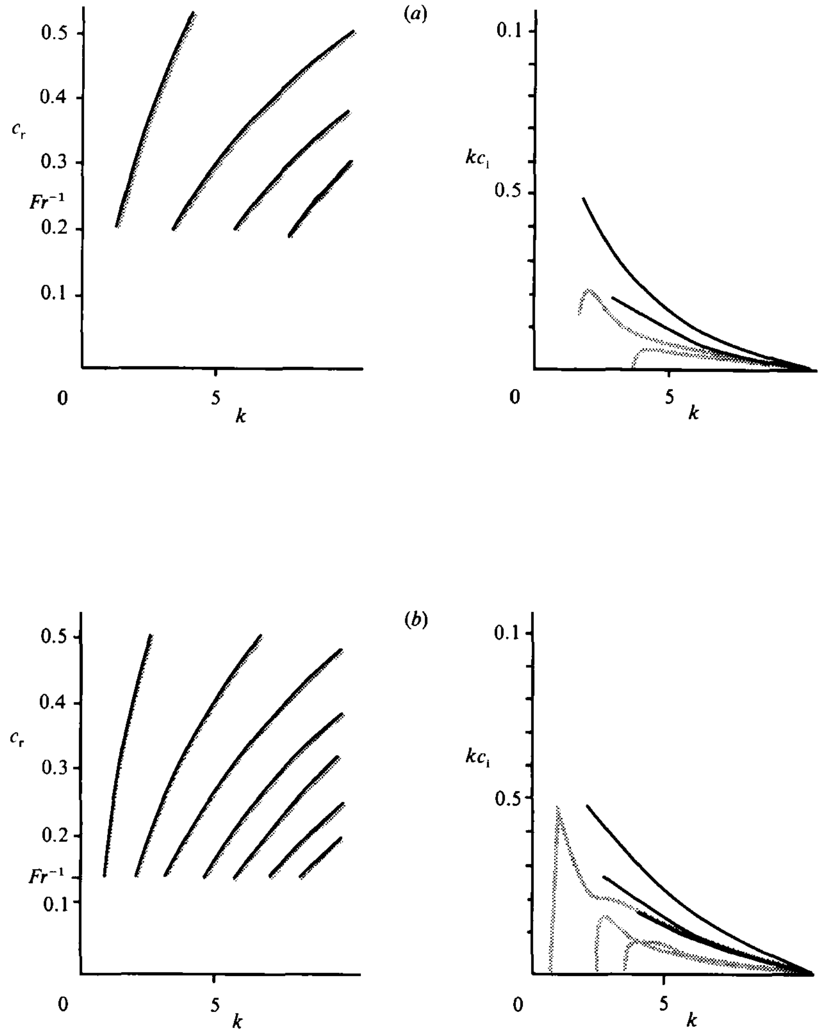

FIGURE 10. Comparison between unstable modes and the quantization and the laser formula. The grey lines are from Case II of Satomura (1981), and the black lines are the estimates by the quantization and the laser formula. (*a*) $Fr = 5.0$, (*b*) $Fr = 7.0$.

by quantization and the Laser Formula. (In these figures, the modes $c_r < 1/Fr$ are not shown. These modes are affected by the break point of the flow (figure 9*a*), so we cannot compare the dispersion relations and the growth rates of these modes with those of the laser formula.) A similar comparison with Satomura's Case I is also possible, although it is a little complicated because of the existence of the two reflecting walls. The growth rates of the unstable modes and those of laser formula show good agreement. Thus, the unstable modes of Satomura (1981) can be described qualitatively with the quantization and laser formula.

5. Ripa's theorem and properties of wave propagation

Ripa (1983) derived the following condition for the stability of flows in a shallow water: if there exists any value of α such that

$$(\alpha - U)\,Q_y \geqslant 0 \quad \text{and} \quad (\alpha - U)^2 \leqslant gH \quad \text{for all} \quad y,$$

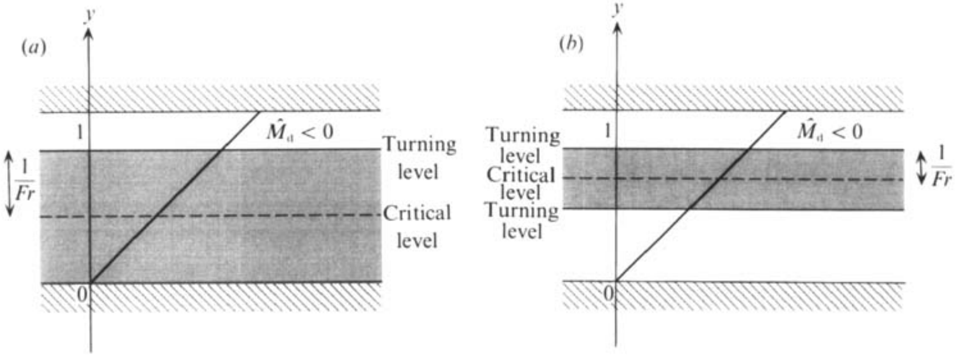

FIGURE 11. Ripa's theorem and properties of wave propagation in Case I of Satomura (1981). The shaded areas indicate the evanescent region. (*a*) The stable case ($Fr < 2$), (*b*) the unstable case ($Fr \geqslant 2$).

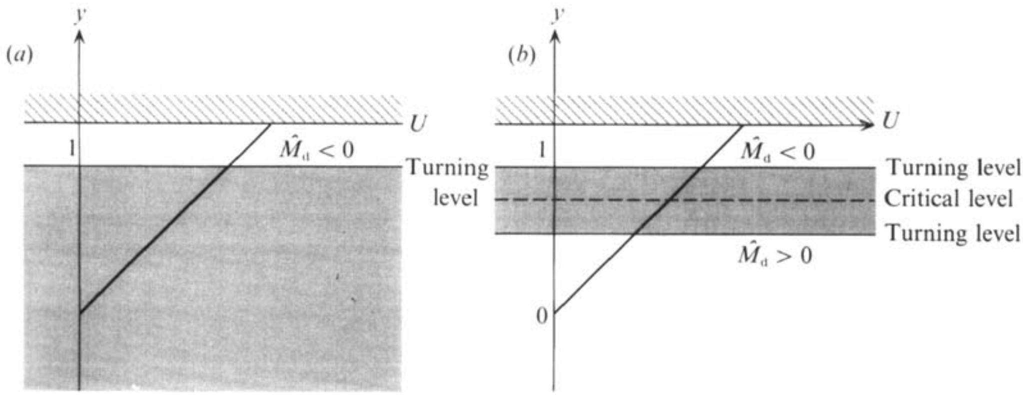

FIGURE 12. Ripa's theorem and properties of wave propagation in Case II of Satomura (1981). The shaded areas indicate the evanescent region. (*a*) The case expected to be stable ($Fr < 2$), (*b*) the unstable case ($Fr \geqslant 2$).

then the flow is stable to infinitesimal perturbations. Q is the potential vorticity of the basic flow.

Let us apply this theorem to the cases of Satomura (1981) and interpret his results in terms of the properties of wave propagation. Since $Q_y = 0$ in Case I of Satomura (1981) (figure 9*a*), the first condition is satisfied automatically. From the second condition, we find that the flow is stable for $Fr \leqslant 2$. In Case II (figure 9*b*), since $Q_y = -\infty$ at the break point where $U = 0$, α should be negative from the first condition, and the flow is stable for $Fr \leqslant 1$ from the second condition.

Recall that as mentioned following (18), the width of the evanescent region near the critical levels is $2/Fr$. In Case I, when $Fr < 2$, that is when there is not unstable mode, the width of the evanescent region is so wide that waves with the same phase speed but with opposite signed $\hat{M}_d$ cannot coexist (figure 11 *a*). On the other hand, when $Fr > 2$, that is when there are unstable modes, the width of the evanescent region is narrow enough, and waves with opposite-signed $\hat{M}_d$ can coexist at the same time (figure 11*b*). This is the condition for the occurrence of over-reflection.

In Case II, the situation becomes a little complicated. From the properties of (divergent) wave propagation, it might be predicted that the flow would be stable for

$Fr \leqslant 2$ since the condition for over-reflection is not satisfied, as shown in figure 12. However, because of the broken profile of the basic flow, Rossby waves exist around there, and the flow becomes unstable even when $1 < Fr < 2$. In this range of Fr, there is no instability caused by the interaction between divergent shallow-water waves.

For the linear shear flow, Ripa's theorem corresponds to the condition that waves with $\hat{M}_d$ of the opposite sign can coexist. That is the condition for over-reflection.

6. Concluding remarks

We have studied the relationship between over-reflection solutions and unstable modes for a linear shear flow of a shallow-water system. The necessary condition for the occurrence of over-reflection of divergent shallow-water waves is that waves with the same phase speed but with $\hat{M}_d$ of the opposite sign can coexist. This corresponds to the conditions for vorticity waves summarized by Lindzen (1988): the critical level must exist; there must be a wave flux sink (wave region) on the opposite side beyond the critical level; the waves can reach the critical level.

The third condition listed above is automatically satisfied because the critical level of shallow-water waves is in an evanescent region. We can interpret the unstable modes of Satomura (1981) by over-reflection solutions which satisfy the quantization condition. The growth rates can be estimated by the laser formula. It was also shown that the integral theorem for the stability derived by Ripa (1983) corresponds to a condition which gives the wave geometry for over-reflection.

The examples of a linear shear flow of shallow water presented in this paper give us the simplest illustration of over-reflection. In the case of Rossby waves (Lindzen & Tung 1978) and internal gravity waves (Lindzen & Barker 1985), there is a momentum flux jump at the critical level, where the wave–mean flow interaction occurs, and then the momentum budget becomes complicated. However, in the case of shallow-water waves considered here, there is no mean flow acceleration and no singularity at the critical level. The interpretation in terms of momentum is quite simple as described in the foregoing section.

Lindzen (1988) pointed out the importance of the solution of Orr (1907) with regard to the mechanism of over-reflection and instability. In Orr's solution, disturbances are amplified when the wave surface is tiled in the direction opposite to the shear, and are reduced when the wave surface is tiled in the direction of the shear. He referred to this tendency as the Orr mechanism, and stated that it is the basic property of disturbances in a shear flow. In the over-reflection solutions which Lindzen and his colleagues have treated, the wave surfaces are tilted in the direction opposite to the shear at the critical levels, which coincides with the amplification phase of Orr's solution. Lindzen (1988) proposed that operation of the amplification phase of the Orr mechanism at the critical level should be the mechanism of over-reflection.

However, the Orr mechanism can only be applied to 'vorticity waves'. Over-reflection of shallow-water waves is not explained by it, because they are 'divergent waves'. In fact, according to the WKBJ approximation, wave packets are reduced when their wave crests are rotated to be vertical, while wave packets are amplified when their crests are rotated to be horizontal. The steady solution shown in figure 6 indicates this tendency. This amplifying and decaying behaviour is opposite to that of Orr's solution.

Lindzen (1988) mentioned that a critical level is essential because it is necessary for the Orr mechanism to operate. But, the Orr mechanism does not operate in the over-reflection of shallow-water waves. A critical level exists simply because waves with the same phase speeds, but with opposite-signed $\hat{M}_d$ can coexist, which is necessary for the momentum to be conserved.

The authors wish to thank Ms Tabata for drawing the figures. Some of the figures were produced by GFD-Dennou library developed by Drs Shiotani and Sakai.

Appendix A. Time integration of shallow-water waves in a linear shear flow

For the time integration of linear equations (10), (12) and (13), we used the following numerical schemes which conserve disturbance momentum:

$$\frac{\partial u_i'}{\partial t} + \mathrm{i}kU_i u_i' + v_i' = -\frac{\mathrm{i}kh_i'}{Fr^2}, \tag{A 1}$$

$$\frac{\partial h_i'}{\partial t} + \mathrm{i}kU_i h_i' + \mathrm{i}ku_i' + \frac{v_{i+1}' - v_{i-1}'}{2\delta y} = 0, \tag{A 2}$$

$$\frac{\partial q_i'}{\partial t} + \mathrm{i}kU_i q_i' = 0, \quad q_i' = \mathrm{i}kv_i' - \frac{u_{i+1}' - u_{i-1}'}{2\delta y} + h_i'. \tag{A 3}$$

Time integrations are executed with the fourth-order Runge–Kutta scheme.

Appendix B. Steady over-reflection solutions in a linear shear flow of shallow water

In this Appendix we solve the scattering problem in a linear shear flow of a shallow-water system. The solution to be obtained is in the form of $\mathrm{e}^{\mathrm{i}k(x-ct)}$. The radiation conditions are applied in the transmitted wave region.

We choose the origin of the y-coordinate at the critical level, $U-c=0$. Equation (14) which is the equation of the Fourier transform of the surface displacement $\hat{h}'$ becomes

$$\frac{\mathrm{d}^2\hat{h}'}{\mathrm{d}y^2} - \frac{2}{y}\frac{\mathrm{d}\hat{h}'}{\mathrm{d}y} + k^2\{Fr^2y^2 - 1\}\hat{h}' = 0. \tag{B 1}$$

We further transform the y-coordinate to $\tilde{y} \equiv ky$, and we have

$$\frac{\mathrm{d}^2\hat{h}'}{\mathrm{d}\tilde{y}^2} - \frac{2}{\tilde{y}}\frac{\mathrm{d}\hat{h}'}{\mathrm{d}\tilde{y}} + \left\{\frac{Fr^2}{k^2}\tilde{y}^2 - 1\right\}\hat{h}' = 0. \tag{B 2}$$

We find from this that the forms of the solutions are determined by only one parameter, Fr/k.

By expanding $\hat{h}'$ around $\tilde{y}=0$ as a power series (Satomura 1981), we have the following two independent solutions:

$$\left.\begin{aligned} \hat{h}_1' &= \sum_{n=0}^{\infty} A_n \tilde{y}^{2n+3}, \quad A_1 = \tfrac{1}{10}A_0, \quad A_{n+1} = \frac{1}{(2n+5)(2n+2)}\left(A_n - \frac{Fr^2}{k^2}A_{n-1}\right). \\ \hat{h}_2' &= \sum_{n=0}^{\infty} B_n \tilde{y}^{2n}, \quad B_1 = -\tfrac{1}{2}B_0, \quad B_{n+1} = \frac{1}{(2n+2)(2n-1)}\left(B_n - \frac{Fr^2}{k^2}B_{n-1}\right). \end{aligned}\right\} \tag{B 3}$$

From (11), (12) and (A 3), the Fourier transforms of velocity $\hat{v}'$, $\hat{u}'$ are written as

$$\hat{v}_1' = -\frac{k}{\mathrm{i}Fr^2}\frac{1}{\tilde{y}}\frac{\partial \hat{h}_1'}{\partial \tilde{y}} = -\frac{k}{\mathrm{i}Fr^2}\sum_{n=0}^{\infty}(2n+3)A_n\tilde{y}^{2n+1},$$

$$\hat{v}_2' = -\frac{k}{\mathrm{i}Fr^2}\frac{1}{\tilde{y}}\frac{\partial \hat{h}_2'}{\partial \tilde{y}} = -\frac{k}{\mathrm{i}Fr^2}\sum_{n=0}^{\infty}2(n+1)B_{n+1}\tilde{y}^{2n},$$

$$\hat{u}_1' = -\frac{\tilde{y}}{k}\hat{h}_1' - \frac{1}{\mathrm{i}}\frac{\partial \hat{v}_1'}{\partial \tilde{y}} = \frac{1}{k}\sum_{n=0}^{\infty}A_n\tilde{y}^{2n+4} + \frac{k}{Fr^2}\sum_{n=0}^{\infty}(2n+3)(2n+1)A_n\tilde{y}^{2n},$$

$$\hat{u}_2' = -\frac{\tilde{y}}{k}\hat{h}_2' - \frac{1}{\mathrm{i}}\frac{\partial \hat{v}_2'}{\partial \tilde{y}} = \frac{1}{k}\sum_{n=0}^{\infty}B_{bn}\tilde{y}^{2n+1} + \frac{k}{Fr^2}\sum_{n=0}^{\infty}4(n+1)(n+2)B_{n+2}\tilde{y}^{2n+1}.$$

Note that $\hat{h}'$, $\hat{u}'$, $\hat{v}'$ have no singularities at the critical level, $\tilde{y} = 0$.

To obtain the reflection and transmission factors, we examine the solution of (B 2) at $\tilde{y} \rightarrow \pm\infty$ (after Lindzen & Baker 1985). By the transformation $\hat{h}' = |\tilde{y}|\tilde{h}'$, (A 2) becomes

$$\frac{\mathrm{d}^2\tilde{h}'}{\mathrm{d}\tilde{y}^2} + \left\{\left(\frac{Fr^2}{k^2}\tilde{y}^2 - 1\right) - \frac{2}{\tilde{y}^2}\right\}\tilde{h}' = 0. \tag{B 4}$$

As $\tilde{y} \rightarrow \pm\infty$, this equation is approximately expressed as

$$\frac{\mathrm{d}^2\tilde{h}'}{\mathrm{d}\tilde{y}^2} + \frac{Fr^2}{k^2}\tilde{y}^2\tilde{h}' = 0. \tag{B 5}$$

By the use of the WKBJ approximation, we have the asymptotic solutions as $\tilde{y} \rightarrow \pm\infty$:

$$\tilde{h}_+' = |P|^{-\frac{1}{4}}\exp\left(\mathrm{i}\int^{\tilde{y}} P^{\frac{1}{2}}\,\mathrm{d}\tilde{y}\right), \tag{B 6}$$

$$\tilde{h}_-' = |P|^{-\frac{1}{4}}\exp\left(-\mathrm{i}\int^{\tilde{y}} P^{\frac{1}{2}}\,\mathrm{d}\tilde{y}\right), \tag{B 7}$$

where $P \equiv (Fr^2/k^2)\tilde{y}^2$. In the above $\tilde{h}_+'$ describes the waves approaching the critical level, while $\tilde{h}_-'$, describes the waves leaving the critical level.

Let us determine the solution which satisfies the boundary conditions. From the radiation condition in the transmitted wave region, we have at $\tilde{y} \rightarrow \infty$

$$\tilde{h}' \equiv \alpha\tilde{h}_1' + \beta\tilde{h}_2' \sim A\tilde{h}_-', \tag{B 8}$$

where $\tilde{h}_{1,2}' \equiv \hat{h}_{1,2}'/|\tilde{y}|$, and α, β, A are complex constants. The ratio of β to α can be determined by using

$$\frac{\mathrm{d}\tilde{h}_-'}{\mathrm{d}\tilde{y}} + \left(\frac{\mathrm{d}P/\mathrm{d}\tilde{y}}{4|P|} + \mathrm{i}P^{\frac{1}{2}}\right)\tilde{h}_-' = 0.$$

It is actually evaluated at a certain point $\tilde{y}_{\mathrm{obs}}$ where the WKBJ approximation is valid:

$$\frac{\beta}{\alpha} = -\left[\frac{\dfrac{\mathrm{d}\tilde{h}_1'}{\mathrm{d}\tilde{y}} + \left(\dfrac{\mathrm{d}P/\mathrm{d}\tilde{y}}{4|P|} + \mathrm{i}P^{\frac{1}{2}}\right)\tilde{h}_1'}{\dfrac{\mathrm{d}\tilde{h}_2'}{\mathrm{d}\tilde{y}} + \left(\dfrac{\mathrm{d}P/\mathrm{d}\tilde{y}}{4|P|} + \mathrm{i}P^{\frac{1}{2}}\right)\tilde{h}_2'}\right]_{\tilde{y}=\tilde{y}_{\mathrm{obs}}}. \tag{B 9}$$

Let us determine the amplitudes of the incident, reflection and transmission waves in order to evaluate the reflection and transmission factors. At $\tilde{y} \to -\infty$, we can express the solution by the incident and reflection waves as follows:

$$\tilde{h}' \sim B\tilde{h}'_{+} + C\tilde{h}'_{-}. \tag{B 10}$$

Differentiating (B 10) with $\tilde{y}$ and using (B 6) and (B 7), we get

$$\frac{\mathrm{d}\tilde{h}'}{\mathrm{d}\tilde{y}} = -\frac{\mathrm{d}P/\mathrm{d}\tilde{y}}{4|P|}\tilde{h}' + \mathrm{i}P^{\frac{1}{2}}(B\tilde{h}'_{+} - C\tilde{h}'_{-}). \tag{B 11}$$

By using (B 10) again, we can evaluate the factors B, C from $\tilde{h}$ as

$$B = \frac{1}{2\mathrm{i}P^{\frac{1}{2}}\tilde{h}'_{+}}\left\{\frac{\mathrm{d}\tilde{h}'}{\mathrm{d}\tilde{y}} + \left(\frac{\mathrm{d}P/\mathrm{d}\tilde{y}}{4|P|} + \mathrm{i}P^{\frac{1}{2}}\right)\tilde{h}'\right\}, \tag{B 12}$$

$$C = -\frac{1}{2\mathrm{i}P^{\frac{1}{2}}\tilde{h}'_{-}}\left\{\frac{\mathrm{d}\tilde{h}'}{\mathrm{d}\tilde{y}} + \left(\frac{\mathrm{d}P/\mathrm{d}\tilde{y}}{4|P|} - \mathrm{i}P^{\frac{1}{2}}\right)\tilde{h}'\right\}. \tag{B 13}$$

Therefore, the reflection factor is

$$R \equiv \left|\frac{C}{B}\right| = \frac{\left|\frac{\mathrm{d}\tilde{h}'}{\mathrm{d}\tilde{y}} + \left(\frac{\mathrm{d}P/\mathrm{d}\tilde{y}}{4|P|} - \mathrm{i}P^{\frac{1}{2}}\right)\tilde{h}'\right|_{\tilde{y}=-\tilde{y}_{\mathrm{obs}}}}{\left|\frac{\mathrm{d}\tilde{h}'}{\mathrm{d}\tilde{y}} + \left(\frac{\mathrm{d}P/\mathrm{d}\tilde{y}}{4|P|} + \mathrm{i}P^{\frac{1}{2}}\right)\tilde{h}'\right|_{\tilde{y}=-\tilde{y}_{\mathrm{obs}}}}. \tag{B 14}$$

In the same manner, we can obtain the factor A of the transmission wave.

$$A = -\frac{1}{2\mathrm{i}P^{\frac{1}{2}}h'_{-}}\left\{\frac{\mathrm{d}\tilde{h}'}{\mathrm{d}\tilde{y}} + \left(\frac{\mathrm{d}P/\mathrm{d}\tilde{y}}{4|P|} - \mathrm{i}P^{\frac{1}{2}}\right)\tilde{h}'\right\}. \tag{B 15}$$

Therefore, the transmission factor T is

$$T \equiv \left|\frac{A}{B}\right| = \frac{\left|\frac{\mathrm{d}\tilde{h}'}{\mathrm{d}\tilde{y}} + \left(\frac{\mathrm{d}P/\mathrm{d}\tilde{y}}{4|P|} - \mathrm{i}P^{\frac{1}{2}}\right)\tilde{h}'\right|_{\tilde{y}=\tilde{y}_{\mathrm{obs}}}}{\left|\frac{\mathrm{d}\tilde{h}'}{\mathrm{d}\tilde{y}} + \left(\frac{\mathrm{d}P/\mathrm{d}\tilde{y}}{4|P|} + \mathrm{i}P^{\frac{1}{2}}\right)\tilde{h}'\right|_{\tilde{y}=-\tilde{y}_{\mathrm{obs}}}}. \tag{B 16}$$

Figure 7 shows the result calculated by choosing $\tilde{y}_{\mathrm{obs}}$ at the distance of two wavelengths away from the turning surface.

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